Computer Vision

MSc Informatics option GVR
James L. Crowley

Fall Semester 20 November 2013

Lesson 6

Scale Invariant Pyramids, HOG, SIFT and Haar

Lesson Outline:

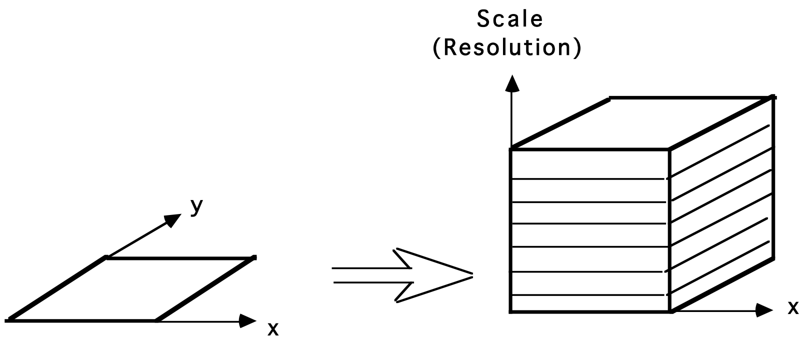
1. Image Scale Space

1.1. Scale Space (Rappel)

Let p(x, y) be an image of size W columns by H rows.
Let G(x, y, σ_s) by a Gaussian function of scale σ_s

Image Scale space is a 3D continuous space *p(x, y, s)*

$$p(x, y, s) = p(x, y) * G(x, y, \sigma_s)$$



Scale space:
Separates global structure from fine detail.
Provides context for recognition.
Can provide local descriptions (features) of the image that are invariant to position, orientation and scale.

Note that the scale axis (s) in scale space is logarithmic

$$s = \mathrm{Log}_2(\sigma) = \mathrm{Log}_2(2^s)$$

A logarithmic scale axis is necessary for scale equivariance.
The appearance of a pattern in the image results in a unique structure in p(x, y, s).
If a shape in an image is made larger by a factor of $D = 2^d$

$$p(x,y) \rightarrow p(x2^d, y2^d)$$

Then the projection of appearance is translated by d in the scale axis

$$p(x, y, s) \rightarrow p(x,y,s+d)$$

Scale space is equivariant in position, scale and rotation

Translate a pattern by Δx, Δy and the structure translates by $x+\Delta x$, $y+\Delta y$ in P(x, y, s).
Rotate by θ in x, y and the structure rotates by θ in P(x, y, s).
Scale by a factor of $2^{\Delta s}$, and the structure translates to $s+\Delta s$ in P(x, y, s).

1.2. Discrete Scale Space - Scale invariant impulse response.

In a computer, we need to discretize (sample) the axes x, y, and s.

Let *p(x, y)* is an image array of size WxH pixels, where (x, y) are integers,

We propose to sample scale with a step size of $\Delta\sigma = 2$ so that $\sigma_k=2^k$
For k=0, to K.

$\sigma_0=1$ is the smallest scale that we can represent.

At k=0 : $\sigma_0=2^0=1$.

let M = min(W, H)
K is the largest scale possible: $K=\text{Log}_2(M)$

at k=K, $\sigma_K=2^K=2^{\log(M)}=M=\min(W,H)$

For k > K the scale parameter σ is larger than the image.

1.3. Spatial Resampling and Image Pyramids

Because the Gaussian, $G(x, y, \sigma_k)$, is a low pass filter, as σ_k grows it becomes possible to resample the image with a larger step size without loss of information.

Such resampling has the benefit of assuring an <u>invariance</u> of the impulse response of each image. The sample size Δx_k, Δy_k can grow exactly as σ_k.

What sample size is possible? It is possible to show that the sample step must be smaller than σ. For example, let $\Delta x = \sigma/2$

Thus $\Delta x_k = \Delta y_k = 2^{k-1}$ with only minimal aliasing.

Resampling selects every Δx image sample:
for integer values of i, j:

$$p(i, j, k) = p(i\Delta x_k, j\Delta y_k, k) = p(x/\Delta x_k, y/\Delta x_k, k)$$

The position in the original image is $x = i\Delta x_k$ and $y = j\Delta y_k$

Resampling at $\Delta x_k = \sigma_k = 2^k$ results an identical impulse response at each level.
This property is called "scale invariance". (The impulse response is scale invariant).

A resampled scale space (a scale invariant pyramid), with a scale step of one "octave".

$$p(i, j, k) = p(x/\Delta x_k, y/\Delta x_k, k) \quad \textit{such that } \Delta x_k = 2^{k+1} \textit{ and } \sigma_k = 2^{k+1}$$

Note that it is possible to build a scale invariant pyramid with a step size of $\Delta\sigma = 2^{k/2}$ however this involves a very complicated resampling scheme that is beyond the scope of this class.

1.4. Using a scale invariant Pyramid to compute image derivatives at scale

Last week we saw that image derivatives can be computed by convolving the image with derivatives of Gaussians

$$p_x(x,y,\sigma) \approx p * G_x(x,y,\sigma)$$

With the Pyramid, derivatives can be obtained directly by sum and difference of the resampled pixels.

Let $i = x/\Delta x_k$ and $j = y/\Delta y_k$

Then

$$p_x(i,j,k) \approx p(i+1,j,k) - p(i-1,j,k)$$
$$p_y(i,j,k) \approx p(i,j+1,k) - p(i,j-1,k)$$
$$p_{xx}(i,j,k) \approx p(i+1,j,k) - 2p(i,j,k) + p(i-1,j,k)$$
$$p_{yy}(i,j,k) \approx p(i,j+1,k) - 2p(i,j,k) + p(i,j-1,k)$$
$$p_{xy}(i,j,k) \approx p(i+1,j+1,k) - p(i-1,j+1,k) - p(i+1,j-1,k) + p(i-1,j-1,k)$$

These are sometimes referred to as "Receptive Fields" because they are similar to the receptive fields observed in the mammalian visual cortex.

Recall the Gradient $\vec{\nabla} P(i,j) = \begin{pmatrix} p_x(i,j) \\ p_x(i,j) \end{pmatrix}$

In a scale-invariant pyramid, the gradient at any sample in the pyramid is

$$\vec{\nabla} p(i,j,k) = \begin{pmatrix} p_x(i,j,k) \\ p_y(i,j,k) \end{pmatrix} = \begin{pmatrix} p(i+1,j,k) - p(i-1,j,k) \\ p(i,j+1,k) - p(i,j-1,k) \end{pmatrix}$$

Laplacien: $\nabla^2 p(x,y,k) = p * \nabla^2 G(x,y,\sigma_k) = p_{xx}(x,y,k) + p_{yy}(x,y,k)$

For a Gaussian Scale Space, we can show that:

$$\nabla^2 G_x(x,y,\sigma) = G_{xx}(x,y,\sigma) + G_{yy}(x,y,\sigma) = \frac{\partial G(x,y,\sigma)}{\partial \sigma}$$

As a consequence: $\nabla^2 G(x, y, \sigma) \approx G(x, y, \sigma_1) - G(x, y, \sigma_2)$

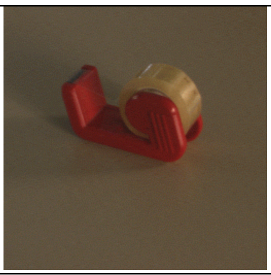
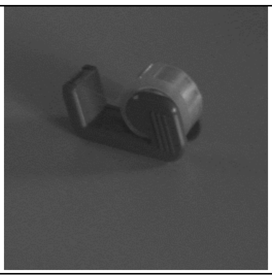
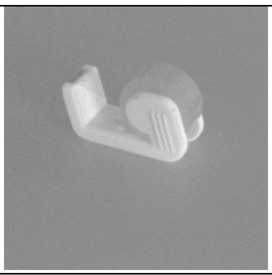
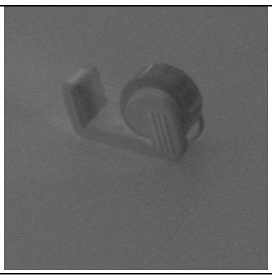
This is called a "Difference of Gaussians" (DoG).
and typically requires $\sigma_1 \geq \sqrt{2}\ \sigma_2$

1.5. Color Opponent Scale Space

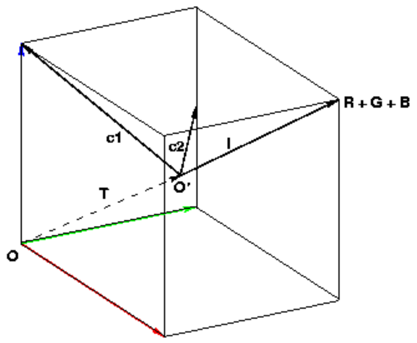
In lesson 3 we saw that a color opponent space was useful for illumination invariance

(R, G, B) ⇒ (L, C_1, C_2)
$$\begin{pmatrix} L \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0.33 & 0.33 & 0.33 \\ -0.5 & -0.5 & 1 \\ 0.5 & -0.5 & 0 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix}$$

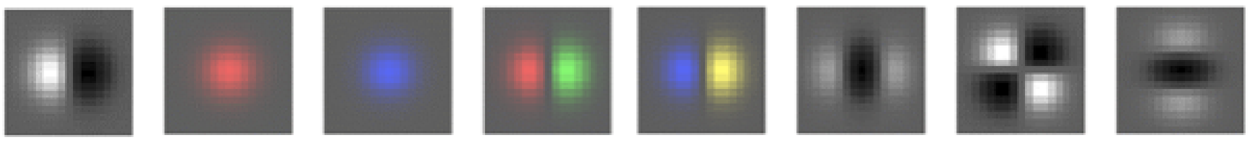
This representation separates luminance and chrominance.

				
RGB		B-W	R-G	R+G-B

Color opponent space can be used to build receptive fields that can be steered in color



$$\begin{pmatrix} L \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0.33 & 0.33 & 0.33 \\ -0.5 & -0.5 & 1 \\ 0.5 & -0.5 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 R \\ \alpha_2 G \\ \alpha_3 B \end{pmatrix}$$



We then compute 3 pyramids : L(x, y, k), C_1(x, y, k), and C_2(x, y, k), This gives us a feature vector for appearance:

$$\vec{A}(x,y,k) = \begin{bmatrix} G_x^{L\sigma_k} \\ G^{C_1\sigma_k} \\ G^{C_2\sigma_k} \\ G_x^{C_1\sigma_k} \\ G_x^{C_2\sigma_k} \\ G_{xx}^{L\sigma_k} \\ G_{xy}^{L\sigma_k} \\ G_{yy}^{L\sigma_k} \end{bmatrix}$$

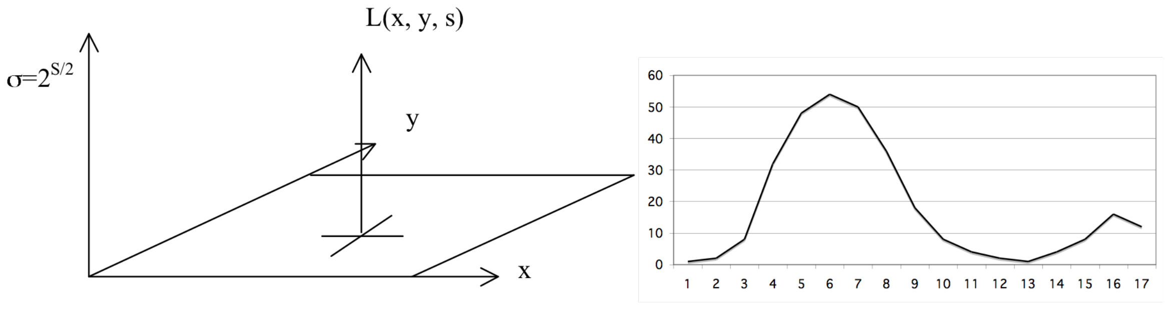
This can be generalized to include multiple scales.

1.6. Intrinsic scale at a point in an image

A Laplacian profile for an image point is the Laplacian of the image computed over a continuous (exponential) range of scales. The pixel position remains constant.

$$L(x, y, s) = p(x, y) * \nabla^2 G(x, y, 2^s)$$

The Laplacian profile is invariant to rotation and translation and equivariant to changes in scale. Since scale is proportional to distance, the profile is equivariant to viewing distance.



A change in viewing distance at x, y shifts the function L(x,y,s) in s.
The function remains the same. Thus the maximum is a local invariant.

The "intrinsic" scale at a point (x, y) is $\sigma_i = 2^{s_i}$

such that $s_i = \arg\text{-}\max_s \{L(x,y,s)\}$

We can do the same with a Gradient.

The Gradient $\vec{\nabla} p(x,y,s) = \begin{pmatrix} p_x(x,y,s) \\ p_y(x,y,s) \end{pmatrix} = \begin{pmatrix} p * G_x(x,y,s) \\ p * G_y(x,y,s) \end{pmatrix}$

For any image point (x,y) the intrinsic scale can be computed from

$$s_i = \arg - \max_s \{ \| \vec{\nabla} p(x,y,s) \| \}$$

These are positions in the image that can serve as landmarks for tracking or recognition.

1.7. Scale Invariant Interest Points from a pyramid

Maximal points in the image derivatives provide landmarks.
These can serve to focus processing, and are thus called "interest points".
In an image scale space, these points are scale invariant. They provide landmarks for scale invariant image description.

Maxima in the Lapacian Scale Space provide Scale invariant interest points

Recall that using an image Pyramid, the Laplacian is simply the difference at adjacent levels.

DoG: $L(i, j, k) = \nabla^2 p(i, j, k) = p(i, j, k) - p(i, j, k-1)$

We can detect scale invariant interest points local maxima in the Laplacian.

$$X(i,j,k) = local - \max_{i,j,k} \{L(i,j,k)\}$$

However, the difference in scale in the pyramid levels is $\Delta\sigma=2$.
We would like to we need a technique to determine intrinsic scale at finer step sizes.

1.8. Natural Interest points at half octave scales.

To obtain a natural interest point with a scale precision of less than $\Delta\sigma=2$ we can use cascade convolution on each pyramid level.

Consider a pyramid image at level k: p(i, j, k) with $\sigma_k = 2^k$

we compute:

$p_1(i, j, k) = p(i, j, k) * G(i, j, 1)$ $\quad \sigma_{k1} = 2^{k+1/2}$

$p_2(i, j, k) = p_1(i, j, k) * G(i, j, 1)$ $\quad \sigma_{k2} = 2^{k+1}$

$p_3(i, j, k) = p_2(i, j, k) * G(i, j, 2)$ $\quad \sigma_{k2} = 2^{k+2}$

$p_4(i, j, k) = p_3(i, j, k) * G(i, j, 4) \qquad \sigma_{k2} = 2^{k+4}$

For each pixel local, we can then calculate 4 Laplacian values. :

$$L_{k0} = p_1(i, j, k) - p(i, j, k)$$
$$L_{k1} = p_2(i, j, k) - p_1(i, j, k)$$
$$L_{k2} = p_3(i, j, k) - p_2(i, j, k)$$
$$L_{k3} = p_4(i, j, k) - p_3(i, j, k)$$

If $L_{k0} < L_{k1} > L_{k2}$ then the point p(i, j, k) is a natural interest point at with $\sigma = 2^{k+1/2}$
If $L_{k1} < L_{k2} > L_{k3}$ then the point p(i, j, k) is a natural interest point with $\sigma = 2^{k+1}$

The position of the natural interest point is $x = i \cdot 2^{k-1}, y = j \cdot 2^{k-1}$

This is the method used to find natural interest points in the SIFT detector described below.

1.9. Other popular interest point detectors.

Other popular detectors for scale invariant interest points include:

Gradient Magnitude: $A(i,j,k) = \underset{i,j,k}{Local-}\max\{\|\vec{\nabla} p(x,y,s)\|\}$

and Determinant of the Hessian: $A(i,j,k) = \underset{i,j,k}{Local-}\max\left\{\det\begin{pmatrix} P_{xx}(i,j,k) & P_{xy}(i,j,k) \\ P_{xy}(i,j,k) & P_{yy}(i,j,k) \end{pmatrix}\right\}$

$$A(i,j,k) = \underset{i,j,k}{Local-}\max\left\{P_{xx}(i,j,k)P_{yy}(i,j,k) - P_{xy}(i,j,k)^2\right\}$$

and the Harris-Laplace.

let $b_2(i,j) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

$$H_x^2 = b_2 * P_x^2$$
$$H_{xy} = b_2 * P_{xy}$$
$$H_y^2 = b_2 * P_y^2$$

$$H = \begin{pmatrix} H_x{}^2 & H_{xy} \\ H_{xy} & H_y{}^2 \end{pmatrix}$$

Harris interest points h(i,j,k) = arg-max{det(H)-Trace(H)}

2. HOG: Histogram of Oriented Gradients

A local histogram of gradient orientation provides a vector of features image appearance that is relatively robust to changes in orientation and illumination.

HOG gained popularity because of its use in the SIFT feature point detector (described next). It was subsequently explored and made popular by Navneet Dalal (M2R GVR 2003) and Bill Triggs.

Recall: The orientation of a gradient at pyramid sample (i,j,k) is:

$$\theta(i,j,k) = Tan^{-1}\left\{\frac{p_y(i,j,k)}{p_x(i,j,k)}\right\}$$

This is a number between 0 and π. We can quantize it to a value between 1 and N value by

$$a(i,j,k) = N \cdot Trunc\left\{\frac{\theta(i,j,k)}{\pi}\right\}+1$$

We can then build a local histogram for a window of size WxH, with upper left corner at i_o, j_o, k. We allocate a table of N cells: h(a). Then for each pixel i,j in our window:

$$\overset{W}{\underset{i=1}{\forall}}\,\overset{H}{\underset{j=1}{\forall}}\, h(a(i+i_o, j+j_o, k)) = h(a(i+i_o, j+j_o, k))+1$$

The result is a local feature composed of N values.
Recall that with histograms, we need around 8 samples per bin to have a low RMS error. Thus a good practice is to have N=W=H. For example N=4, W=4 and H=4. Many authors ignore this and use values such as N=8, W=4, H=4, resulting in a sparse histogram.

Remark: A fast version when N=4 replaces the inverse tangent by computing the diagonal derivatives with differences:

$$P_{\frac{\pi}{4}}(i,j,k) = P(i+1,j+1,k) - P(i-1,j-1,k)$$
$$P_{\frac{\pi}{2}}(i,j,k) = P(i,j+1,k) - P(i,j-1,k)$$
$$P_{\frac{3\pi}{4}}(i,j,k) = P(i+1,j-1,k) - P(i-1,j+1,k)$$
$$P_{\pi}(i,j,k) = P(i+1,j,k) - P(i-1,j,k)$$

To determine a(i,j,k) simply choose the maximum.

3. Scale Invariant Feature Transform (SIFT)

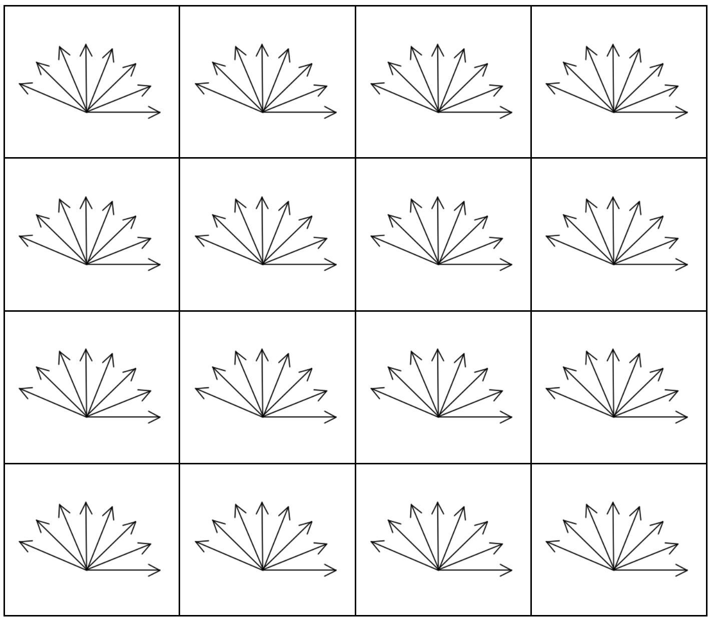
SIFT uses a scale invariant pyramid to compute scale invariant interest points

$$X(i,j,k) = \underset{i,j,k,R=2}{Local - \max}\{P(i,j,k) - P(i,j,k-1)\}$$

For each interest point, it then computes a U x V grid of HOG detectors with N=8, W=4, H=4 at the level k
Typically U=V=4.

At level k, $\Delta i = \Delta j = 2^{k/2}$

This gives 16 x 16 = 128 features at each interest point.
This feature vector is invariant to changes in position and scale and very robust with changes in image plane rotation and illumination intensity.



Various authors experiment with other grid sizes.

For example, let the grid size be G.

G=4, W=4, H=4, N=4

Gives 64 features.

4. Fast 2D Haar Wavelets using Integral Image

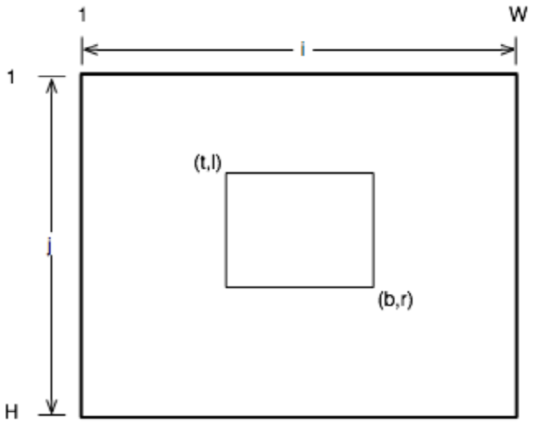
In 2001, Paul Viola and Mike Jones at MERL (Misubishi Research Labs) showed that Haar wavelets could be used for real time face detection using a cascade of linear classifiers.

They computed the Haar Wavelets (difference of adjacent boxes) for a window from integral images.

4.1. Difference of Boxes

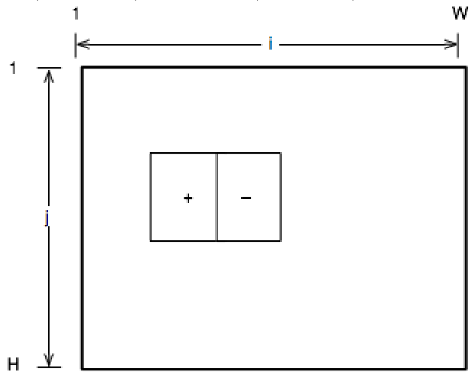
A box feature is a sum of pixel from (t, l) to (b, r)
With the constraints : t < b and r > l.

$$b(t,l,b,r) = \sum_{x=l}^{r}\sum_{y=t}^{b} p(x,y)$$



A first order Difference of Boxes (DoB) feature is a difference of two boxes $box(t1,l1,b1,r1)$.

$$DoB(t1,l1,b1,r1,t2,l2,b2,r2) = box(t1,l1,b1,r1) - box(t2,l2,b2,r2)$$

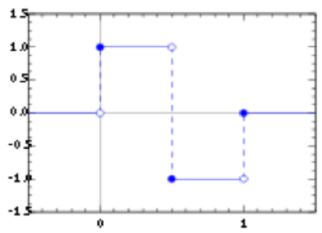


An interesting subclass are Difference of Adjacent Boxes where the sum of pixels is 0. These are Haar wavelets. They can be computed for an image, or for an extracted window of an image (an "imagette").

4.2. Haar Wavelets

Haar A. Zur Theorie der orthogonalen Funktionensysteme, Mathematische Annalen, 69, pp 331–371, 1910.

The Haar wavelet is a difference of rectangular Windows.



The Digital (discrete sampled) form of Haar wavelet is

$$h(n;d,k) = \begin{cases} 1 & \text{for } \mathrm{d} \le n < d + k/2 \\ -1 & \text{for } \mathrm{d} + \mathrm{k}/2 \le n < d + k \\ 0 & \text{for } n < d \text{ and } n \ge d + k \end{cases}$$

Haar wavelets can be used to define an orthogonal transform analogous to the Fourier basis. This can be used to define an orthogonal transform (the Walsh-Hadamard Transform). The basis is

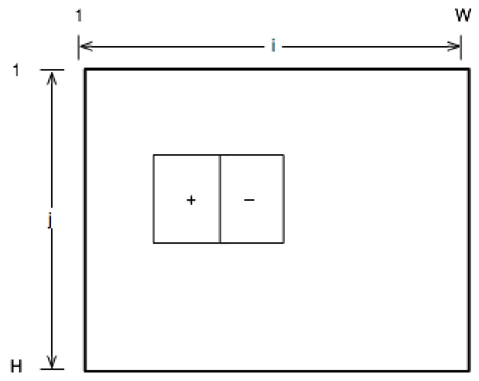
$$H_0 = +1 \qquad H_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \cdots$$

$$H_m = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{bmatrix}$$

Haar Functions, and the Walsh-Hadamard transform have been used in Functional Analysis and signal processing for nearly a century.

In the 1980s the Wavelet community re-baptized the Haar functions as "wavelets" and demonstrated that the Walsh-Hadamard transform is the simplest form of wavelet transform.

A 2-D form of Walsh-Hadamard transform may be defined using DoB features using adjacent boxes. These can be calculated VERY fast using an algorithm known as Integral Images. They give a VERY large number of possible image features.



Assume a window is extracted from an image and mapped to the WxH imagette. Label the window coordinates (x, y) from [1,W] and [1,H]

Parameters:
1) The "polarity" of the difference ([1 -1] or [-1 1])
1) order (number of adjacent boxes): 2nd or 3rd
2) orientation: vertical or horizontal
3) center position - (c_x, c_y) WxH possible positions
4) box size (d_x, d_y) (W/2)x(H/2) possible sizes

These can provide N image features. Label these with an integer index, *n*, $H_n(x,y)$
Note that each Haar wavelet corresponds to a specific position, size, and orientation in the imagette.

The product of each Haar wavelet $H_n(x,y)$ with the imagette $W(x,y)$ gives a number: X_n. This number is an image "feature" that describes the imagette.

$$X_n = \sum_{x=1}^{W} \sum_{y=1}^{H} W(x,y) H_n(x,y)$$

Given a WxH imagette of a face we can obtain N Feature numbers, X_n. Not all features are useful. We will use "machine learning to determine the subset of useful features for detecting faces.

Do not be confused by the reuse of W and H. W and H are the size of the imagette, W(x,y) is the imagette and $H_n(x,y)$ are the

4.3. Integral Images

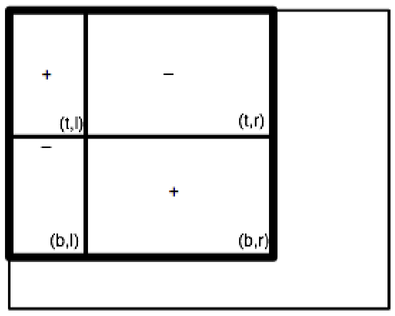
An integral image is an image where each pixel contains the sum from the upper left corner:

$$ii(u,v) = \sum_{i=1}^{u}\sum_{j=1}^{v} W(i,j)$$

An integral image provides a structure for very fast computation of 2D Haar wavelets.

Any box feature can be computed with 4 operations (additions/subtractions).

$$box(t,l,b,r)=ii(b,r)–ii(t,r)–ii(b,l)+ii(t,l)$$

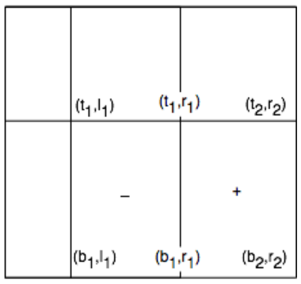


An arbitrary 1st order difference of boxes costs 8 ops.

$$DoB(t_1,l_1,b_1,r_1,t_2,l_2,b_2,r_2) = box(t_1,l_1,b_1,r_1)–box(t_2,l_2,b_2,r_2)$$
$$=ii(b_1,r_1)–ii(t_1,r_1)–ii(b_1,l_1)+ii(t_1,l_1) – (ii(b_2,r_2)–ii(t_2,r_2)–ii(b_2,l_2)+ii(t_2,l_2))$$

However, a 1st order Haar wavelet costs only 6 ops because $r_1=l_2$ and thus

$$ii(t_1,r_1) = ii(t_2,l_2) \text{ and } ii(b_1,r_1)= ii(b_2,l_2)$$



$$Haar(t_1,l_1,b_1,r_1,b_2,r_2) = ii(b_2,r_2)–2ii(b_1,r_1)+ii(b_1,l_1)–ii(t_2,r_2)+2ii(t_1,r_1)–ii(t_1,l_1)$$

4.4. Fast Integral Image algorithm.

Integral images have been used for decades to compute local energy for normalization of images. A fast recursive algorithm for computing the integral image makes use of a buffer, c(i). The buffer keeps a running sum of each column.

```
For j = 1 to H
For i = 1 to W
{       c(i) = c(i) +  p(i,j)
        ii(i,j) = ii(i-1,j) + c(i) }
```

Cost = 2WH ops.